Lecture 2

The Semi Empirical Mass Formula

SEMF

Overview

- 1 The liquid drop model
- 2 The Coulomb Term
- 3 Mirror nuclei, charge asymmetry and independence
- 4 The Volume and Surface Terms
- 5 The asymmetry term
- 6 The pairing term
- 7 The SEMF

0 Introduction to the SEMF

- Aim: phenomenological understanding of nuclear binding energies as function of A, Z and N.
- Assumptions:
  - Nuclear density is constant (see lecture 1).
  - We can model effect of short range attraction due to strong interaction by a liquid drop model.
  - Coulomb corrections can be computed using electro magnetism (even at these small scales)
  - Nucleons are fermions at T=0 in separate wells (Fermi gas model → asymmetry term)
  - QM holds at these small scales → pairing term.
- Compare with experiment: success & failure!

1 Liquid Drop Model Nucleus

- Phenomenological model to understand binding energies.
- Consider a liquid drop
  - Ignore gravity and assume no rotation
  - Intermolecular force repulsive at short distances, attractive at intermediate distances and negligible at large distances ➔ constant density.
  - n=number of molecules, T=surface tension, B=binding energy
    E=total energy of the drop, α,β=free constants

$$E=-\alpha n + 4\pi R^2 T \quad \rightarrow \quad B=\alpha n-\beta n^{2/3}$$

surface area ~ $n^{2/3}$

- Analogy with nucleus
  - Nucleus has constant density
  - From nucleon-nucleon scattering experiments we know:
    - Nuclear force has short range repulsion and is attractive at intermediate distances.
  - Assume charge independence of nuclear force, neutrons and protons have same strong interactions ➔check with experiment (Mirror Nuclei!)

2 Coulomb Term

- The nucleus is electrically charged with total charge *Ze*
- Assume that the charge distribution is spherical and compute the reduction in binding energy due to the Coulomb interaction

$$E_{Coulomb} = \int_0^{Ze} \frac{Q(r)}{4\pi\varepsilon_0 r} dQ \qquad Q(r) = Ze(r/R)^3 \quad dQ = 3Zer^2/R^3 dr$$

to change the integral to *dr ; R*=outer radius of nucleus

$$E_{Coulomb} = \int_0^R \frac{3(Ze)^2}{4\pi\varepsilon_0 r} \frac{r^5}{R^6} dr = (3/5)\frac{(Ze)^2}{4\pi\varepsilon_0 R}$$

includes self interaction of last proton with itself. To correct this replace Z^2 with $Z*(Z-1)$

… and remember $R=R_0A^{-1/3}$

$$B_{Coulomb}(Z,A) = -d\frac{Z*(Z-1)}{A^{1/3}}$$

in principle you could take d from this calculation but it is more accurate to take it from the overall fit of the SEMF to data (nuclei not totally spherical or homogeneous)

3 Mirror Nuclei

- Does the assumption of the drop model of constant binding energy for every constituent of the drop acatually hold for nuclei?
- Compare binding energies of mirror nuclei (nuclei with n←→p). Eg ${}^{7}_{3}Li$ and ${}^{7}_{4}Be$.
- If the assumption holds the mass difference should be due to n/p mass difference and Coulomb energy alone.
- Let's compute the Coulomb energy correction from results on previous page

$$\Delta E_{coulomb}(Z,Z-1) = \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0 R}[Z(Z-1)-(Z-1)(Z-2)] = \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0 R}2(Z-1)$$

$Z \sim A/2\,;\, R = R_0 A^{1/3}$ to find that $\Delta E_C(Z,Z-1) \propto A^{2/3}$

- Now lets measure mirror nuclei masse, assume that the model holds and derive $\Delta E_{Coulomb}$ from the measurement.
- This should show an $A^{2/3}$ dependence
- And the scaling factor should yield the correct R_0 of 1.2 fm
- if the assumptions were right

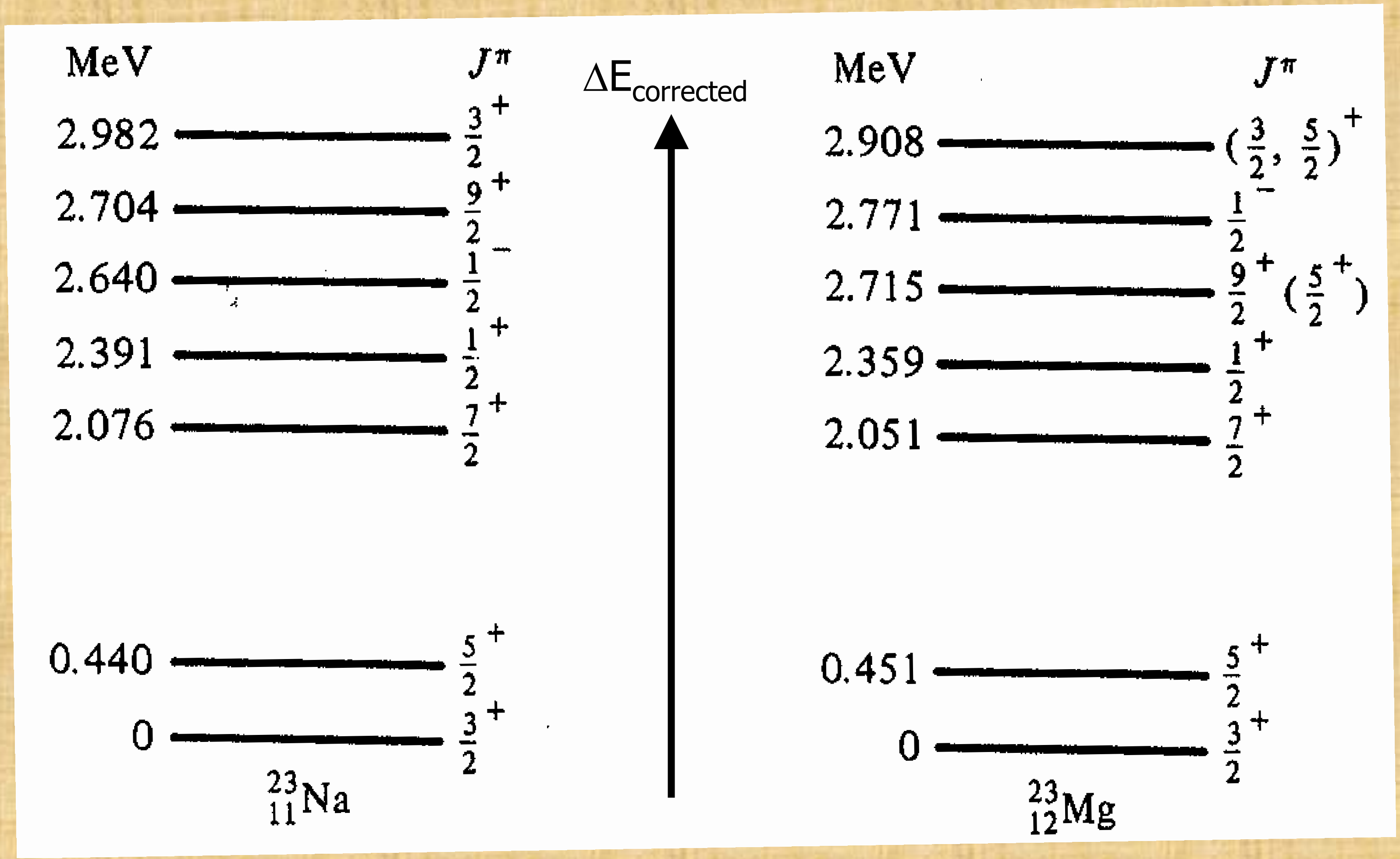
Charge symmetry

nn and **pp** interaction same (apart from Coulomb)

Figure 3.10 Coulomb energy differences of mirror nuclei. The data show the expected $A^{2/3}$ dependence, and the slope of the line gives $R_0 = 1.22$ fm.

3 More charge symmetry

- Energy Levels of two mirror nuclei for a number of excited states
- Corrected for n/p mass difference and Coulomb Energy



3 From Charge Symmetry to Charge Independence

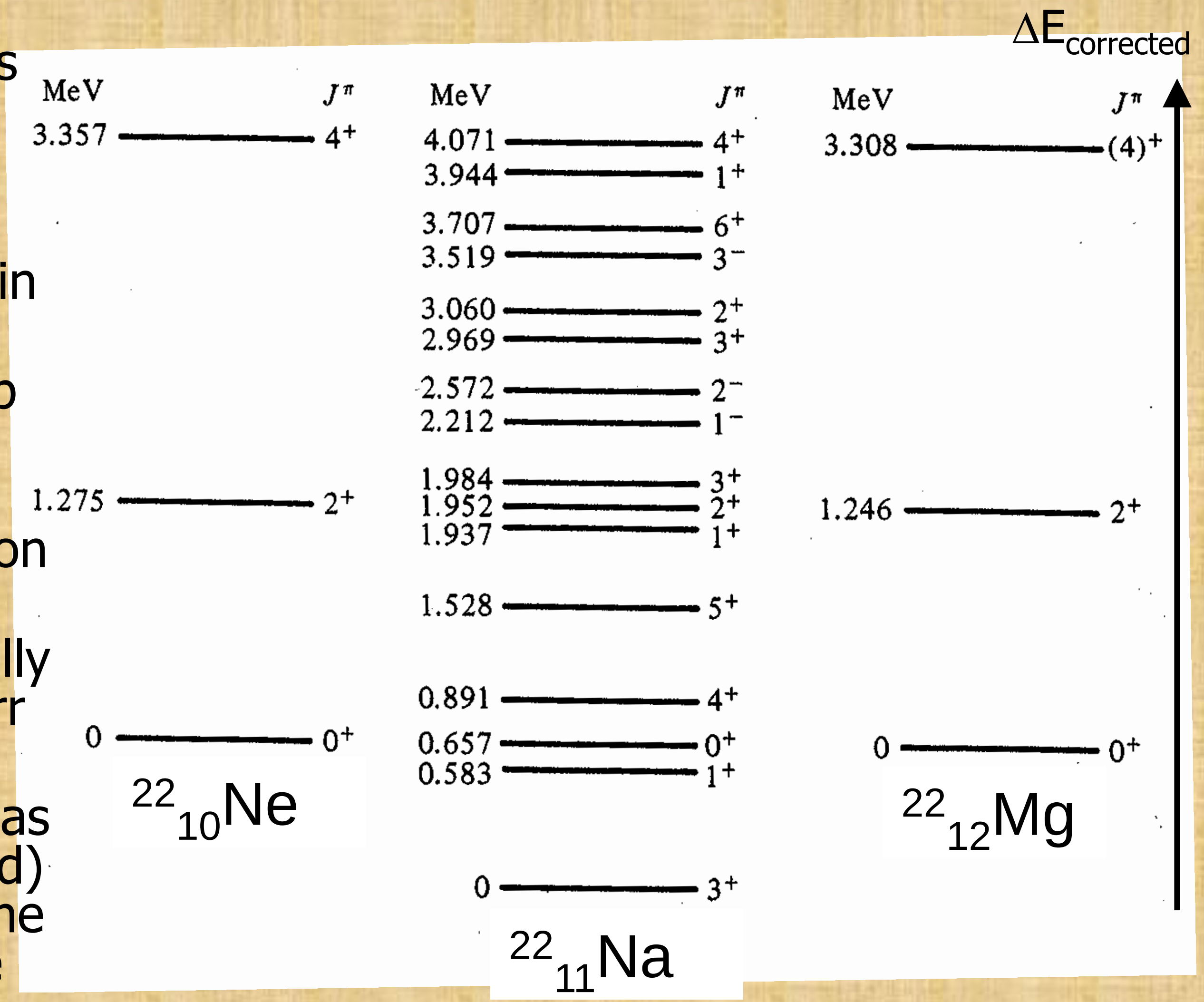
- Mirror nuclei showed that strong interaction is the same for nn and pp.
- What about np ?
- Compare energy levels in "triplets" with same A, different number of n and p. e.g.

$$^{22}_{10}\mathrm{Ne} \qquad ^{22}_{11}\mathrm{Na} \qquad ^{22}_{12}\mathrm{Mg}$$

- If we find the same energy levels for the same spin states ➔ Strong interaction is the same for np as nn and pp.
- See next slide

3 Charge Independence

- Same spin/parity states should have the same energy.
- Yes: np=nn=pp
- Note: Far more states in $^{22}_{11}Na$. Why?
- Because it has more np pairs then the others
- np pairs can be in any Spin-Space configuration
- pp or nn pairs are excluded from the totally symmetric ones by Herr Pauli
- Note also that $^{22}_{11}Na$ has the lowest (most bound) state, remember for the deuteron on next page



3 Charge Independence

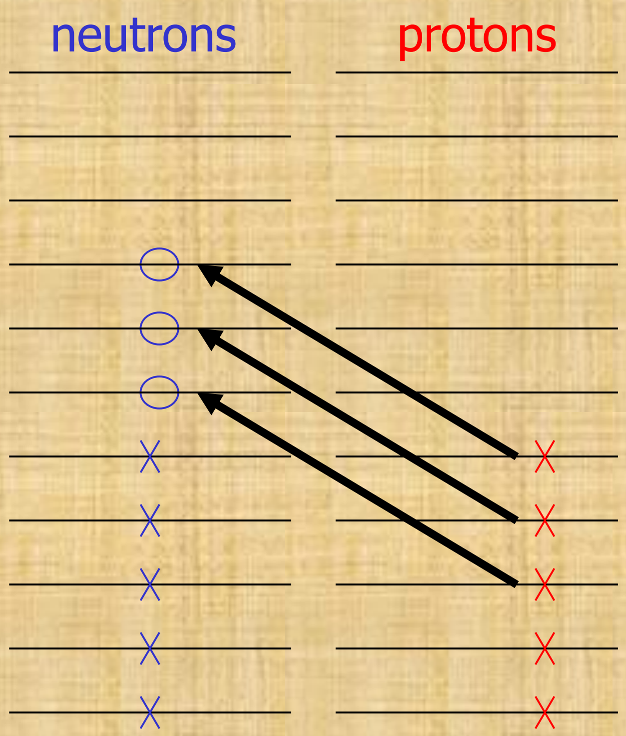
- We have shown by measurement that:
  - If we correct for n/p mass difference and Coulomb interaction, then energy levels in nuclei are unchanged under n ←→ p
  - and we must change nothing else! I.e. spin and space wavefunctions must remain the same!
- Conclusion: strong two-body interaction same for pp, pn and nn if nucleons are in the same quantum state.
- Beware of the Pauli exclusion principle! eg why do we have bound state of pn but not pp or nn?
  - because the strong force is spin dependent and the most strongly bound spin-space configurations (deuteron) are not available to nn or pp. It's Herr Pauli again!
  - Just like $^{22}_{11}$Na on the previous triplet level schema

4 Volume and Surface Term

- We now have all we need to trust that we can apply the liquid drop model to a nucleus
  - constant density
  - same binding energy for all constituents
- Volume term: $B_{Volume}(A) = +aA$
- Surface term: $B_{Surface}(A) = -bA^{2/3}$
- Since we are building a phenomenological model in which the coefficients a and b will be determined by a fit to measured nuclear binding energies we must inlcude any further terms we may find with the same A dependence together with the above

5 Asymmetry Term

- Neutrons and protons are spin ½ fermions ➔ obey Pauli exclusion principle.
- If all other factors were equal nuclear ground state would have equal numbers of n & p.



Illustration

- n and p states with same spacing Δ.
- Crosses represent initially occupied states in ground state.
- If three protons were turned into neutrons
- the extra energy required would be $3 \times 3\ \Delta$.
- In general if there are Z-N excess protons over neutrons the extra energy is $((Z-N)/2)2\ \Delta$. relative to Z=N.
- But how big is Δ ?

5 Asymmetry Term

- Assume:
  - p and n form two independent, non-interacting gases occupying their own square Fermi wells
  - $kT << \Delta$
  - so we can neglect kT and assume $T=0$
  - This ought to be obvious as nuclei don’t suddenly change state on a warm summers day!
  - Nucleons move non-relativistically (check later if this makes sense)

5 Asymmetry Term

- From stat. mech. density of states in 6d phase space = $1/h^3$

$$dN_{particle} = \frac{4\pi p^2 dpV}{h^3}$$ here $N_{particle}$ could be the number of protons or neutrons

- Integrate up to p_f to get total number of protons Z (or Neutrons N), & Fermi Energy (all states filled up to this energy level).

$$Z = (4\pi V / 3h^3)\ p_F{}^3 \text{ and } V = \frac{4}{3}\pi R_0^3 A$$

$$P_F = (3/4\pi)^{2/3} \frac{h}{R_0}\left(\frac{Z}{A}\right)^{1/3} \text{ and } E = p^2/2m \Rightarrow E_F = (3/4\pi)^{4/3}\frac{h^2}{2mR_0^2}\left(\frac{Z}{A}\right)^{2/3}$$

- Change variables p ➔ E to find avg. E

$$dN/dE = \frac{dN/dp}{dp/dE} = const \cdot E^{1/2} \qquad <E> = \frac{\int_0^{E_F} E^{3/2} dE}{\int_0^{E_F} E^{1/2} dE} = (3/5)E_F$$

These are all standard stat. mech. results!

5 Asymmetry Term

- Compute total energy of all protons by *Z*<E>* — call this K

$$E^P_{Total} = Z\left\langle E^P \right\rangle = \frac{3}{5}(3/4\pi)^{4/3}\frac{h^2}{2mR_0^2} Z\left(\frac{Z}{A}\right)^{2/3}$$

- Use the above to compute total energy of Z protons and N neutrons

$$E_{Total} = \frac{K}{A^{2/3}}\left[Z^{5/3} + N^{5/3}\right]$$

change variables from (Z,N,A) to (y,A) with y=N-Z

$$E_{Total} = \frac{KA^{5/3}}{A^{2/3}2^{5/3}}\left[(1-y/A)^{5/3} + (1+y/A)^{5/3}\right]$$

where y/A is a small number (ε)

$$E_{Total} = \frac{KA}{2^{5/3}}\left[(1-\varepsilon)^{5/3} + (1+\varepsilon)^{5/3}\right]$$

- Binomial expansion keep lowest term in y/A

$$(1-\epsilon)^{5/3} = 1 - \frac{5}{3}\epsilon + \frac{5}{3}\frac{2}{3}\epsilon^2 + O(\epsilon^3)$$
$$(1+\epsilon)^{5/3} = 1 + \frac{5}{3}\epsilon + \frac{5}{3}\frac{2}{3}\epsilon^2 + O(\epsilon^3)$$

note! linear terms cancel

$$E_{Total} = 2^{-2/3}KA + \frac{2^{1/3}5}{9}K\frac{(N-Z)^2}{A}$$

This terms is only proportional to volume (A). It has already been captured by the Volume term of the liquid drop model

$$\Delta E_{Total}(Fermi-Gas) = const * \frac{(N-Z)^2}{A}$$

comes from a fit of the SEMF to measurements
analytical ≈ 24 MeV

5 Asymmetry term

- From the Fermi Gas model we learn that
  - due to the fermionic nature of p and n we loose in binding energy if the nucleus deviates from N=Z

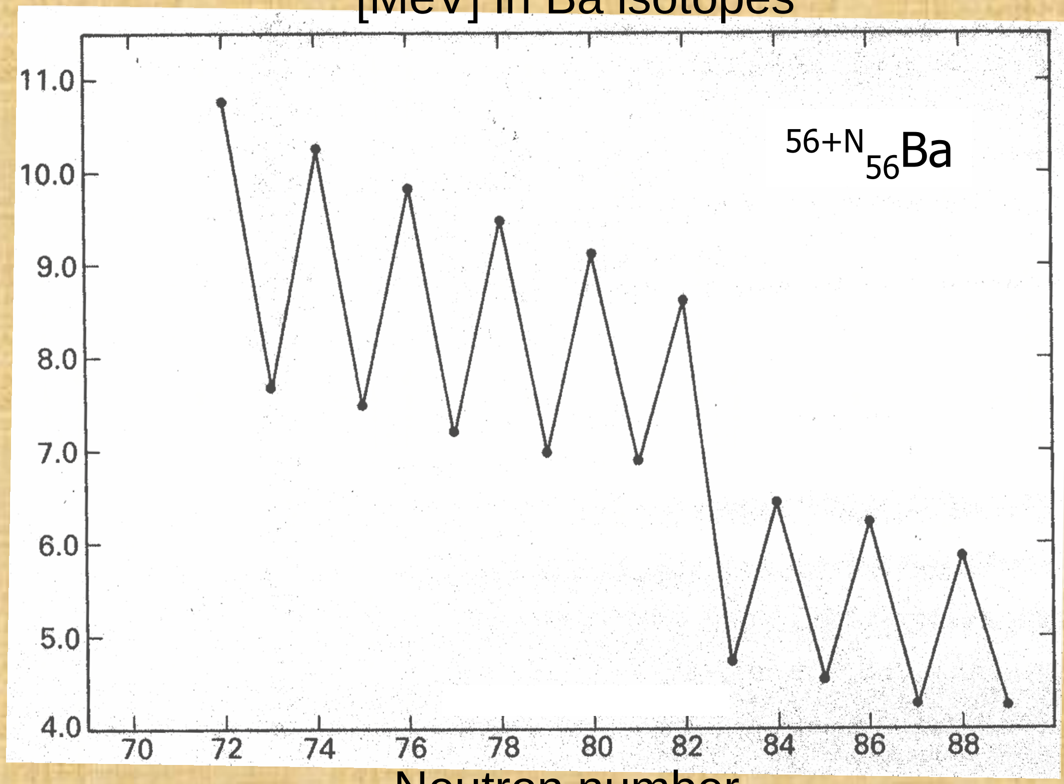
- The Asymmetry term: $B_{Asymmetry}(N,Z) = -c\frac{(N-Z)^2}{A}$

6 Pairing Term

- Observations:
- Nuclei with even number of n or even number of p more tightly bound then with odd numbers. See figure
- Only 4 stable o-o nuclei but 153 stable e-e nuclei.
- p energy levels are Coulomb shifted wrt n ➔ small overlap of wave functions between n and p.
- Two p or two n in same energy level with opposite values of j_z have AS spin state
  - ➔ forced into sym spatial w.f.
  - ➔ maximum overlap
  - ➔ maximum binding energy because of short range attraction.

Note: this only holds for nn and pp, not for np. ➔ We don't have a preference for even A

Neutron separation energy [MeV] in Ba isotopes



Neutron number

6 Pairing Term

- Measure that the Pairing effect smaller for larger A
- Phenomenological*) fit to A dependence gives $A^{-1/2}$

$$B_{Pairing}(A) = -\frac{\delta}{A^{1/2}}$$

	δ
e-e	+ive
e-o	0
o-o	-ive

Note: If you want to plot binding energies versus A it is often best to use odd A only as for these the pairing term does not appear

*) For an even more insightful explanation of the A dependence read the book by Jelley

6 Semi Empirical Mass Formula

- Put everything together:

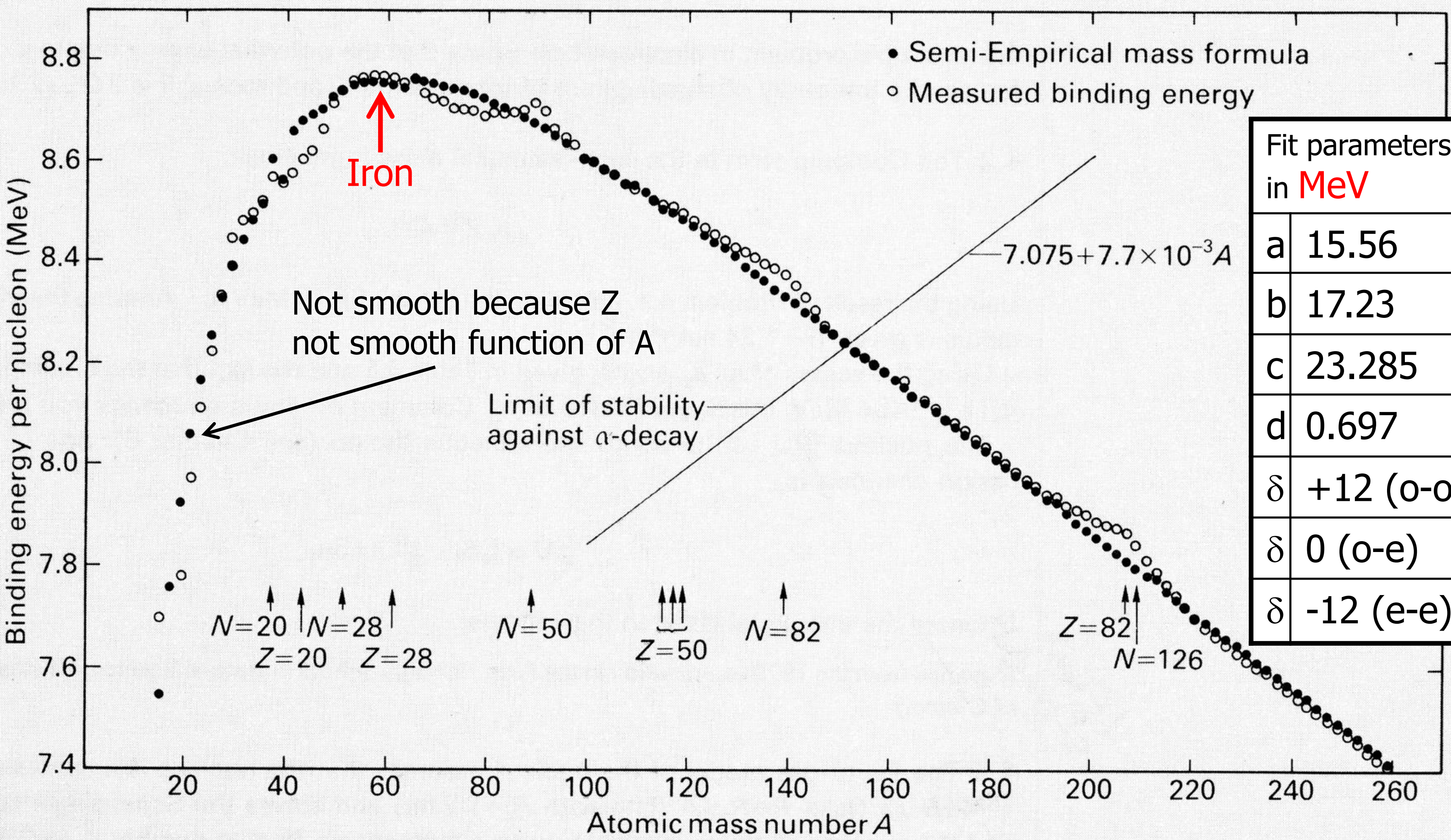
$$B(N,Z) = aA - bA^{2/3} - c\frac{(N-Z)^2}{A} - d\frac{Z^2}{A^{1/3}} - \frac{\delta}{A^{1/2}}$$

Volume Term — Surface Term — Asymmetry Term — Coulomb Term — Pairing Term

- Lets see how all of these assumptions fit reality
- And find out what the constants are
  - Note: we went back to the simpler Z^2 instead of $Z*(Z-1)$

6 Semi Empirical Mass Formula

Binding Energy vs. A for beta-stable odd-A nuclei



Fit parameters in MeV

a	15.56
b	17.23
c	23.285
d	0.697
δ	+12 (o-o)
δ	0 (o-e)
δ	-12 (e-e)

6 Semi Empirical Mass Formula

- Conclusions
  - Only makes sense for $A \geq 20$
  - Good fit for large A (good to <1%) in most places.
  - Deviations are interesting ➔ shell effects.
  - Coulomb term constant agrees with calculation.
  - Explains the valley of stability (see next lecture).
  - Explains energetics of radioactive decays, fission and fusion.